Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

3. Inverse. If w = f[z] is any transformation that has an inverse, prove the fact that f and its inverse have the same fixed points.

```
Clear["Global`*"]
```

This may be a pud proof, but... I assume that w=f[x] has at least one fixed point. However many it has, consider an arbitrary fixed point z_1 in z-plane which gets mapped to w-plane by w=f[z], to a point called w_1. And the inverse of w, w^{-1}, a mapping in its own right, takes the point w_1 and operates on it to map it to the z-plane, to the exact point where it originated, that being the inverse function's function. Now $w_1 = z_1$ because z_1 is a fixed point for w. By observing w^{-1} as it maps w_1 to z_1, with which it is equal, I can see that w_1 is a fixed point for w^{-1}. The establishment of the fixed point in w^{-1} is separate and independent of the establishment of z_1 as a fixed point for w, yet, as if I didn't already know, I can look at z_1 and w_1 and see that they are equal. Therefore the mapping functions have a fixed point that is the same, and if they have this arbitrary one, they share all in common.

5. Derive the mapping in Example 2 from numbered line (2) on p. 746.

```
Clear["Global`*"]
```

$z_1 = 0$

0

$z_2 = 1$

1

Mathematica cannot swallow ∞ assigned to a variable in this context. For now, it needs to be symbolic, as below. (Note: the w-z assignment formula has its own means of dealing with occurrences of ∞, but with Mathematica I can take care of it a different way.)

$z_3 = a$

a

$w_1 = -1$

-1

$w_2 = -i$

$-i$

$w_3 = 1$

1

$$\text{Solve}\left[\frac{w - w_1}{w - w_3}\left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{z - z_1}{z - z_3}\left(\frac{z_2 - z_3}{z_2 - z_1}\right),\ \{w,\ z\}\right]$$

Solve::svars : Equations may not give solutions for all "solve" variables ≫

$$\left\{\left\{w \to \frac{-i\, a - (1 - i)\, z + a\, z}{i\, a - (1 + i)\, z + a\, z}\right\}\right\}$$

$$\text{Limit}\left[\frac{-i\, a - (1 - i)\, z + a\, z}{i\, a - (1 + i)\, z + a\, z},\ a \to \infty\right]$$

$$\frac{-i + z}{i + z}$$

The expression in the green cell above matches the answer to example 2 on p. 748.

8 - 16 LFTs from three points and images

Find the LFT that maps the given three points onto the three given points in the respective order.

9. 1, i, -1 onto i, -1, $-i$

```
Clear["Global`*"]
```

$z_1 = 1$

1

$z_2 = i$

i

$z_3 = -1$

-1

$w_1 = i$

i

$w_2 = -1$

-1

$w_3 = -i$

$-i$

$$\text{Solve}\left[\frac{w - w_1}{w - w_3}\left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{z - z_1}{z - z_3}\left(\frac{z_2 - z_3}{z_2 - z_1}\right),\ \{w\}\right]$$

$\{\{w \to i\, z\}\}$

11. - 1, 0, 1 onto - i, -1, i

```
Clear["Global`*"]
```

$z_1 = -1$

-1

$z_2 = 0$

0

$z_3 = 1$

1

$w_1 = -i$

$-i$

$w_2 = -1$

-1

$w_3 = i$

i

$$\text{Solve}\left[\frac{w - w_1}{w - w_3}\left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{z - z_1}{z - z_3}\left(\frac{z_2 - z_3}{z_2 - z_1}\right), \{w\}\right]$$

$$\left\{\left\{w \to \frac{i + z}{-i + z}\right\}\right\}$$

13. 0, 1, ∞ onto ∞, 1, 0

```
Clear["Global`*"]
```

$z_1 = 0$

0

$z_2 = 1$

1

$z_3 = a$

a

$w_1 = a$

a

$w_2 = 1$

1

$w_3 = 0$

0

$$\text{Solve}\left[\frac{w - w_1}{w - w_3}\left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{z - z_1}{z - z_3}\left(\frac{z_2 - z_3}{z_2 - z_1}\right), \{w\}\right]$$

$$\left\{\left\{w \to \frac{a - z}{1 - 2 z + a z}\right\}\right\}$$

$$\text{Limit}\left[\frac{a - z}{1 - 2 z + a z}, a \to \infty\right]$$

$$\frac{1}{z}$$

15. 1, i, 2 onto 0, $-i - 1$, $-\frac{1}{2}$

```
Clear["Global`*"]
```

$z_1 = 1$

1

$z_2 = i$

i

$z_3 = 2$

2

$w_1 = 0$

0

$w_2 = -i - 1$

$-1 - i$

$w_3 = -\frac{1}{2}$

$-\frac{1}{2}$

$$\text{Solve}\left[\frac{w - w_1}{w - w_3}\left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{z - z_1}{z - z_3}\left(\frac{z_2 - z_3}{z_2 - z_1}\right), \{w\}\right]$$

$$\left\{\left\{w \to \frac{1 - z}{z}\right\}\right\}$$

17. Find an LFT that maps $|z| \leq 1$ onto $|w| \leq 1$ so that $z = \frac{i}{2}$ is mapped onto $w = 0$. Sketch the images of the lines x = const and y=const.

```
Clear["Global`*"]
```

Numbered line (3) on p. 749 is part of an example that works this problem out for me, except it is left in general form, the z-plane point z_0 being mapped to the origin of the w-plane, while the z-unit-circle is mapped to the w-unit-circle.

$w = \frac{z - z_0}{c\,z\; - 1}$, $c = z_0{}^*$, Abs$[z_0] < 1$

One caveat here is that of the need to tailor c, the conjugate of z_0, to the specific z_0 chosen.

$z_0 = \frac{i}{2}$

$\frac{i}{2}$

zcon = $z_0{}^*$

$-\frac{i}{2}$

w[z_] = $\frac{z - \frac{i}{2}}{\text{zcon}\; z - 1}$

$\frac{-\frac{i}{2} + z}{-1 - \frac{i z}{2}}$

Simplify[%]

$\frac{1 + 2\,i\,z}{-2\,i + z}$

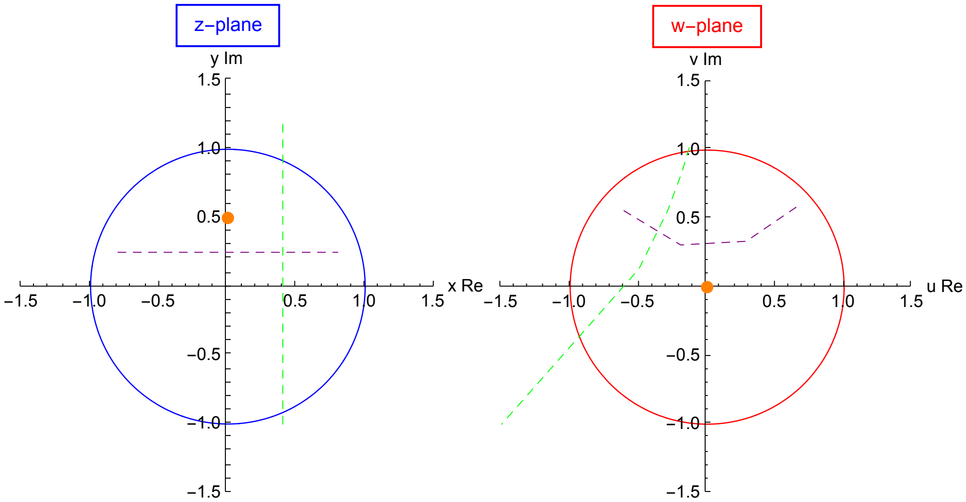
The cell below demonstrates that the green cell agrees with the text answer in content.

PossibleZeroQ$\left[\frac{1 + 2\,i\,z}{-2\,i + z} - \frac{2\,z - i}{-(i\,z + 2)}\right]$

True

The transformation w maps the point z=0+$\frac{i}{2}$ on the z-plane to w=0+0i on the w-plane. In the z-plane plot its location is consistent with the vertical axis location, and in the w-plane plot it is shown correctly.

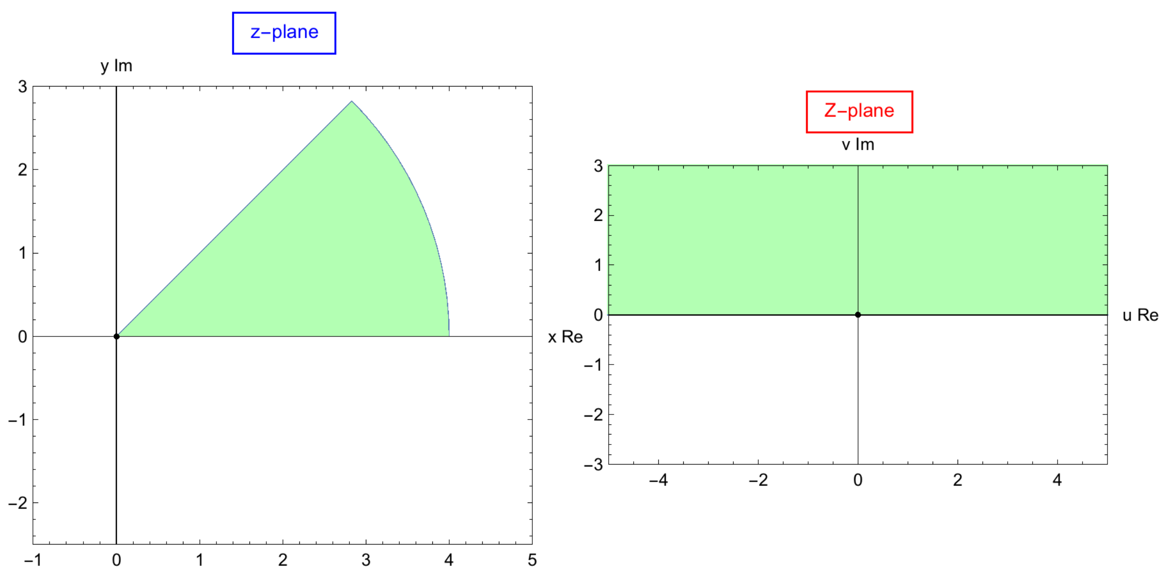
As for the original and transformed circles, and constant lines, the mapping w is allowed to do its work directly wherever possible. Two intersecting test lines demonstrate that intersection angles are preserved under conformal mapping.



19. Find an analytic function $w = f[z]$ that maps the region $0 \leq \mathrm{Arg}[z] \leq \frac{\pi}{4}$ onto the unit disk $|w| \leq 1$.

```
Clear["Global`*"]
```

A hint from s.m. points me to example 2 on p. 739. This explains how to map a wedge sector onto an upper half plane. The upshot is that a wedge sector $0 \leq \;\leq \frac{\pi}{n}$ can be mapped to the upper half plane v ≥ 0 using the transform z^n. In this problem I have a sector from 0 to $\frac{\pi}{4}$, so n =4. As far as the r value goes, it can be anything above about 2, but it can't be ∞. I assume that setting it at 4 arbitrarily will not undermine the argument that the demo is general.



As the plot windows above show, the mapping scheme is successful so far. The next phase is to figure out how to map the upper half plane to a unit circle on the w-plane. The strategy used by the s.m., which I will follow, will be to use the mapping formula advanced in problem 5 and following. To do this I need to pick three points on Z-plane and three others on w-plane and then calculate the expression to do it. As suggesed by s.m., I choose -1, 0, and 1 on the u axis of Z-plane, and intend to map them to (-1, $-i$, and 1), on the unit circle, in the w-plane.

```
Z1 = -1
-1

Z2 = 0
0

Z3 = 1
1

w1 = -1
-1

w2 = -i
-i

w3 = 1
1
```

$$\texttt{exp1 = Simplify}\left[\texttt{Solve}\left[\frac{w - w_1}{w - w_3}\left(\frac{w_2 - w_3}{w_2 - w_1}\right) == \frac{Z - Z_1}{Z - Z_3}\left(\frac{Z_2 - Z_3}{Z_2 - Z_1}\right), \{w\}\right]\right]$$

$$\left\{\left\{w \to \frac{1 + i Z}{i + Z}\right\}\right\}$$

To re-wind this mapping back to the original sector, I make a substitution,

$$\texttt{exp2 = exp1 /. } Z \to z^4$$

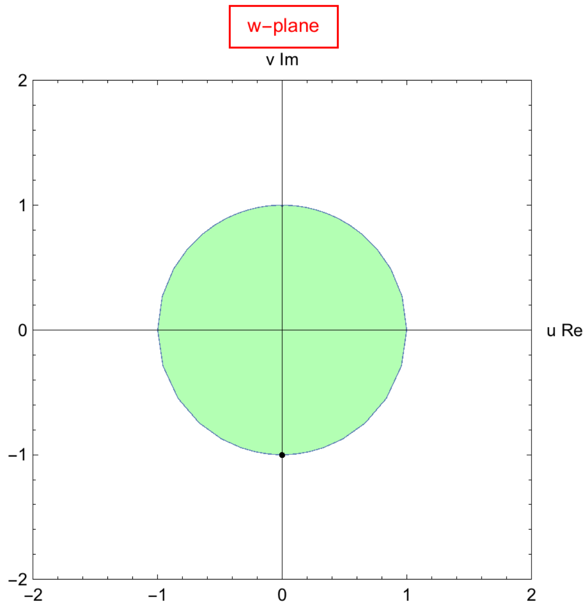
$$\left\{\left\{w \to \frac{1 + i z^4}{i + z^4}\right\}\right\}$$

Since this does not look exactly like the text answer, I check its equivalence, then confer the green.

$$\texttt{PossibleZeroQ}\left[\frac{1 + i z^4}{i + z^4} - \frac{(z^4 - i)}{(-i z^4 + 1)}\right]$$

```
True
```

Time to set up the final mapping transit, taking points from the z-plane pie sector to the w-plane unit circle.



The origin point of the z-plane ends up at the bottom of the unit circle in the w-plane.